次の不定積分を求めよ。

(1) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$ (2) $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$

(3) $\displaystyle\int \frac{x^3}{\sqrt{x-1}}\,dx$ (4) $\displaystyle\int \frac{x^5}{\sqrt{x^2-1}}\,dx$

〔基本問題〕

(1) $\sqrt{x-1}=t$ とおくと

$t^2=x-1$ $x=t^2+1$ より $dx=2t\,dt$

$\therefore$ 与式 $=\displaystyle\int \frac{t^2+1}{t}\cdot 2t\,dt=\int 2t^2+2\,dt$

$=\frac{2}{3}t^3+2t+C$

$\therefore$ $\frac{2}{3}(x-1)\sqrt{x-1}+2\sqrt{x-1}+C$

整理して $\frac{2}{3}(x+2)\sqrt{x-1}+C$

(2) $\sqrt{x+1}=t$ とおくと

$t^2=x+1$ $x=t^2-1$ $dx=2t\,dt$ より

与式 $=\displaystyle\int \frac{2t}{t(t^2-1)}\,dt=\int \frac{2}{t^2-1}\,dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$

$\therefore$ $\log|t-1|-\log|t+1|+C$

$\therefore$ $\log\left|\frac{t-1}{t+1}\right|+C$

ゆえに $\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$

(3) $\sqrt{x-1} = t$ とおくと

$x-1 = t^2 \qquad x = t^2+1$

$dx = 2t\,dt$

$$与式 = \int \frac{(t^2+1)^3}{t} 2t\,dt = 2\int (t^2+1)^3 dt$$

$$= 2\int (t^6+3t^4+3t^2+1)\,dt$$

$$= 2\left(\frac{1}{7}t^7+\frac{3}{5}t^5+t^3+t\right)+C$$

$$= 2\left\{\frac{1}{7}(x-1)^3\sqrt{x-1}+\frac{3}{5}(x-1)^2\sqrt{x-1}+(x-1)\sqrt{x-1}+\sqrt{x-1}\right\}+C$$

$$= 2\sqrt{x-1}\left(\frac{1}{7}x^3-\frac{3}{7}x^2+\frac{3}{7}x-\frac{1}{7}+\frac{3}{5}x^2-\frac{6}{5}x+\frac{3}{5}+x\cancel{-1}\cancel{+1}\right)+C$$

$$= \frac{2}{35}\sqrt{x-1}\,(5x^3-15x^2+15x-5+21x^2-42x+21+35x)+C$$

$$= \frac{2}{35}(5x^3+6x^2+8x+16)\sqrt{x-1}+C$$

(4) $\sqrt{x^2-1} = t$ より $\quad t^2 = x^2-1 \quad x^2 = t^2+1 \quad 2t\,dt = 2x\,dt$ より $t\,dt = x\,dx$

$$\int \frac{(t^2+1)^2}{t}t\,dx = \int (t^4+2t^2+1)\,dt$$

$$= \frac{1}{5}t^5+\frac{2}{3}t^3+t+C$$

$$= \frac{1}{5}(x^2-1)^2\sqrt{x-1}+\frac{2}{3}(x^2-1)\sqrt{x^2-1}+\sqrt{x^2-1}+C$$

$$= \frac{1}{15}(3x^4+4x^2+8)\sqrt{x^2-1}$$