次の不定積分を求めよ。

(1) $\displaystyle\int x^2\cos 2x\,dx$ (2) $\displaystyle\int e^x\cos^2 x\,dx$

(3) $\displaystyle\int \sin^4 x\,dx$ (4) $\displaystyle\int \cos^4 x\,dx$

〔基本問題〕

(1) $\displaystyle\int \overset{f(x)}{x^2}\,\overset{g'(x)}{\cos 2x}\,dx = x^2\cdot\frac{1}{2}\sin 2x - \int 2x\cdot\frac{1}{2}\sin 2x\,dx$

$x\sin 2x$ $\quad -\frac{1}{2}\cos 2x$

$$= \frac{x^2}{2}\sin 2x - \left(x\cdot -\frac{1}{2}\cos 2x + \int\frac{1}{2}\cos 2x\,dx\right)$$

$$= \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$$

$$= \frac{1}{4}\sin 2x(2x^2-1) + \frac{x}{2}\cos 2x + C$$

(2) $\displaystyle\int e^x\cos^2 x\,dx$ $\qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \quad \therefore \cos^2 x = \frac{\cos 2x + 1}{2}$

$$= \int e^x\cdot\frac{\cos 2x+1}{2}\,dx = \frac{1}{2}\left\{\int e^x\cos 2x\,dx + \int e^x\,dx\right\} \quad \cdots ①$$

ここで $\displaystyle\int e^x\cos 2x = e^x\cdot\frac{1}{2}\sin 2x - \int\frac{e^x}{2}\sin 2x\,dx$

$$= \frac{e^x}{2}\sin 2x - \left(\frac{e^x}{2}\cdot -\frac{1}{2}\cos 2x + \int\frac{e^x}{4}\cos 2x\,dx\right)$$

$$\int e^x\cos 2x = \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x - \int\frac{e^x}{4}\cos 2x\,dx$$

$$\frac{5}{4}\int e^x\cos 2x = \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x$$

$$\int e^x\cos 2x = \frac{1}{5}(2e^x\sin 2x + e^x\cos 2x)$$

$\therefore$ ①は

$$\frac{1}{2}\cdot\frac{1}{5}(2e^x\sin 2x + e^x\cos 2x) + \frac{1}{2}e^x + C$$

ゆえに

$$\frac{1}{5}e^x\sin 2x + \frac{1}{10}e^x\cos 2x + \frac{1}{2}e^x + C$$

3C 積分54 2/2

$\cos 2x = 1 - 2\sin^2 x$

$\cos^2 x - \sin$ 　 $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$1-$

(3) $\displaystyle\int \sin^4 x\,dx = \int \sin^3 x \sin x\,dx$

$\displaystyle = \sin^3 x \cdot (-\cos x) - \int 3\sin^2 x \cdot \cos x \cdot (-\cos x)\,dx$

$\displaystyle = -\sin^3 \cos x + \int 3\sin^2 x(1-\sin^2 x)\,dx$

$\displaystyle = -\sin^3 \cos x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$

$\displaystyle \therefore 4\int \sin^4 x\,dx = -\sin^3 \cos x + 3\int \frac{1-\cos 2x}{2}\,dx$

$\displaystyle = -\sin^3 x \cos x + \int \frac{3}{2}\,dx - \frac{3}{2}\int \cos 2x\,dx$

$\cos^2 x - 1 +$

$\displaystyle = -\sin^3 x \cos x + \frac{3}{2}x - \frac{3}{2}\cdot\frac{1}{2}\sin 2x$

ゆえに $\displaystyle\int \sin^4 x\,dx = \frac{-\sin^3 x\cos x}{4} + \frac{3x}{8} - \frac{3\sin 2x}{16} + C$

(4) $2\cos^2 x = \cos 2x + 1$ 　 $\therefore \cos^2 x = \dfrac{\cos 2x + 1}{2}$

$\dfrac{\cos 4x + 1}{2}$

$\displaystyle\int \cos^4 x\,dx = \frac{1}{4}\int(\cos 2x + 1)^2\,dx = \frac{1}{4}\int(\cos^2 2x + 2\cos 2x + 1)\,dx$

$\displaystyle = \frac{1}{4}\int\left(\frac{\cos 4x}{2} + \frac{1}{2} + 2\cos 2x + 1\right)dx$

$\displaystyle = \frac{1}{4}\left(\frac{1}{8}\sin 4x + \frac{1}{2}x + \sin 2x + x\right) + C$

$\displaystyle \therefore \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + C$

別解

(3) $\displaystyle \sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$\displaystyle = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$

$\displaystyle \therefore$ 与式 $\displaystyle = \frac{1}{8}\int(-4\cos 2x + 3 + \cos 4x)\,dx$

$\displaystyle = \frac{1}{8}\left(-2\sin 2x + 3x + \frac{1}{4}\sin 4x\right) + C$

したがって

$\displaystyle \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + C$

いつしょなんだろうね… 変形で至ぱい orz