$\cancel{Q}$'08

正整数 n に対して，

$$a_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \cdots + \frac{1}{2n+(2n-1)}$$

とおく。$\lim_{n\to\infty} a_n$ を求めよ。 〔群馬大〕

a_n をつぎのようにして考える.

$$a_n = \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \frac{1}{2n+3} + \frac{1}{2n+4} + \cdots + \frac{1}{2n+(2n-1)} + \frac{1}{2n+2n}\right)$$
$$- \left(\frac{1}{2n+2} + \frac{1}{2n+4} + \frac{1}{2n+6} + \frac{1}{2n+8} + \cdots + \frac{1}{2n+2n}\right)$$

$$= \sum_{k=1}^{2n} \frac{1}{2n+k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{n+k}$$

$$= \frac{1}{2n}\sum_{k=1}^{2n} \frac{1}{1+\frac{k}{2n}} + \frac{1}{2n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$$

$$\therefore \lim_{n\to\infty} a_n = \int_0^1 \frac{1}{1+x}\,dx - \frac{1}{2}\int_0^1 \frac{1}{1+x}\,dx$$

$$= \Big[\log|1+x|\Big]_0^1 - \frac{1}{2}\Big[\log|1+x|\Big]_0^1$$

$$= \log 2 - \frac{1}{2}\log 2$$

$$= \log\frac{2}{\sqrt{2}}$$

$$= \log\sqrt{2}$$

$$= \frac{1}{2}\log 2$$