①942

(1) 関数 $y = x\sqrt{x^2+1} + \log(x + \sqrt{x^2+1})$ を微分せよ。

(2) 曲線 $x = t^3,\ y = t^2 - t^3\ (0 \leqq t \leqq 1)$ の長さを求めよ。

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(1)

$$y = x(x^2+1)^{\frac{1}{2}} \quad \log\{x + (x^2+1)^{\frac{1}{2}}\}$$

$$y' = (x^2+1)^{\frac{1}{2}} + x \cdot \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}} \cdot \cancel{2}x + \frac{1 + \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}} \cdot \cancel{2}x}{x + \sqrt{x^2+1}}$$

$$= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{\sqrt{x^2+1} + x}{x\sqrt{x^2+1} + x^2 + 1}$$

$$\left(ここで\ \frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1} = \frac{\cancel{x+\sqrt{x^2+1}}}{\sqrt{x^2+1}\,(\cancel{x+\sqrt{x^2+1}})} = \frac{1}{\sqrt{x^2+1}}\ より \right)$$

$$= \frac{x^2+1}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{2(x^2+1)}{\sqrt{x^2+1}}$$

$$= \underline{2\sqrt{x^2+1}}$$

(2)

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 2t - 3t^2$$

よって求める長さLは

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{9t^4 + (2t-3t^2)^2}\,dt$$

$$= \int_0^1 \sqrt{18t^4 - 12t^3 + 4t^2}\,dt$$

$$= \int_0^1 t\sqrt{18t^2 - 12t + 4}\,dt = \sqrt{2}\int_0^1 t\sqrt{(3t-1)^2+1}\,dt$$

$3t - 1 = s$ とおくと $3dt = ds$ 　 $t = \frac{s+1}{3}$

t	$0 \to 1$
s	$-1 \to 2$

よって

$$\sqrt{2}\int_{-1}^{2} \frac{s+1}{3} \cdot \sqrt{s^2+1} \cdot \frac{1}{3}\,ds = \frac{\sqrt{2}}{9}\int_{-1}^{2} (s+1)\sqrt{s^2+1}\,ds$$

$$= \frac{\sqrt{2}}{9}\left\{\int_{-1}^{2} s\sqrt{s^2+1}\,ds + \int_{-1}^{2} \sqrt{s^2+1}\,ds\right\}$$

(1)より

$$= \frac{\sqrt{2}}{9}\left[\frac{1}{3}(s^2+1)^{\frac{3}{2}}\right]_{-1}^{2} + \frac{\sqrt{2}}{9} \cdot \frac{1}{2}\left[x\sqrt{x^2+1} + \log(x + \sqrt{x^2+1})\right]_{-1}^{2}$$

$$= \frac{\sqrt{2}}{27}(5\sqrt{5} - 2\sqrt{2}) + \frac{\sqrt{2}}{18}\{2\sqrt{5} + \log(2+\sqrt{5}) + \sqrt{2} + \log(\sqrt{2}-1)\}$$

$$= \frac{8\sqrt{10} - 1}{27} + \frac{\sqrt{2}}{18}\log(\sqrt{2}+1)(2+\sqrt{5}) \quad \cdots (答)$$