無限級数 5

次の無限級数の和を求めよ。

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)(2n+1)}$

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+3n}$

(4) $\displaystyle\sum_{n=1}^{\infty}\log_2\frac{n+2}{n+1}$

〔練習問題〕

(1) $\displaystyle\lim_{n\to\infty}\frac{n^2}{(2n-1)(2n+1)}=\infty$ より発散

(2) $\displaystyle\frac{1}{\sqrt{n}+\sqrt{n+1}}=\frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)}=\sqrt{n+1}-\sqrt{n}$

$\displaystyle S_n=\sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}$

$=\cancel{\sqrt{2}}-1+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\cancel{\sqrt{4}}-\cancel{\sqrt{3}}+\sqrt{5}-\cancel{\sqrt{4}}\cdots+\cancel{\sqrt{n}}-\cancel{\sqrt{n-1}}+\sqrt{n+1}-\cancel{\sqrt{n}}$

$=-1+\sqrt{n+1}$ これから

$\displaystyle\lim_{n\to\infty}S_n=\infty$ より 発散

(3) $\displaystyle\frac{1}{n(n+3)}$ より $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+3n}=\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right)$ となるので

$\displaystyle S_n=\frac{1}{3}\left\{\left(1-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{5}}\right)+\left(\frac{1}{3}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{8}}\right)\cdots\right.$

$\displaystyle\left.\cdots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)+\left(\cancel{\frac{1}{n-1}}-\frac{1}{n+2}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+3}\right)\right\}$

よって $\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{3}\left\{1+\frac{1}{2}+\frac{1}{3}-\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)\right\}$

$\displaystyle=\frac{11}{18}$

(4) $\displaystyle S_n=\sum_{n=1}^{\infty}\log_2\frac{n+2}{n+1}=\log_2\frac{3}{2}+\log_2\frac{4}{3}+\log_2\frac{5}{4}+\cdots\log_2\frac{n+2}{n+1}$

$\displaystyle=\log_2\left(\frac{\cancel{3}}{2}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{5}{\cancel{4}}\cdots\frac{n+2}{\cancel{n+1}}\right)=\log_2\frac{n+2}{2}$ となる

$\displaystyle\therefore\lim_{n\to\infty}S_n=\lim_{n\to\infty}\log_2\frac{n+2}{2}$

$\displaystyle=\lim_{n\to\infty}\log(n+2)-1=\infty$ よって発散