無限級数 7

数列 $\{a_n\}$ において，$a_n = \dfrac{n+1}{1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)}$ $(n = 1, 2, 3, \cdots)$ であるとき，

(1) $S_n = a_1 + a_2 + \cdots\cdots + a_n$ を簡単にせよ。

(2) $\lim_{n\to\infty} S_n$ を求めよ。

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(1)
$$\sum_{k=1}^{n} k(k+1) = \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$
$$= \frac{1}{12}n(n+1)\{2(2n+1)+6\} = \frac{1}{12}n(n+1)(4n+8)$$
$$= \frac{1}{3}n(n+1)(n+2) \text{ より}$$

$$a_n = \frac{3(n+1)}{n(n+1)(n+2)} = \frac{3}{n(n+2)}$$

$$\underline{a_n = \frac{3}{n(n+2)}}$$

(2)
$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{3}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$
$$= \frac{3}{2}\left\{\left(1-\cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)\right\}$$
$$= \frac{3}{2}\left\{1 + \frac{1}{2} - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right\}$$

よって
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left\{1 + \frac{1}{2} - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right\}$$
$$= \frac{9}{4}$$