

三角関数18

$\sin\theta + \cos\theta = \frac{1}{2}$ のとき, $\tan\theta - \frac{1}{\tan\theta}$ は $-\frac{\boxed{\text{ア}}\sqrt{\boxed{\text{イ}}}}{\boxed{\text{ウ}}}$ である。ただし, $0^\circ < \theta < 180^\circ$ とする。 [明治大]

$$\begin{aligned} (\sin\theta + \cos\theta)^2 &= \left(\frac{1}{2}\right)^2 \\ 2\sin\theta\cos\theta + 1 &= \frac{1}{4} \\ 2\sin\theta\cos\theta &= -\frac{3}{4} \\ \sin\theta\cos\theta &= -\frac{3}{8} \quad \text{①} \end{aligned}$$

$$\sin\theta + \cos\theta = \frac{1}{2} \quad \text{②}$$

$$\text{②より } \cos\theta = \frac{1}{2} - \sin\theta$$

$$\sin\theta \left(\frac{1}{2} - \sin\theta\right) = -\frac{3}{8}$$

$$-\sin^2\theta + \frac{1}{2}\sin\theta = -\frac{3}{8}$$

$$8\sin^2\theta - 4\sin\theta - 3 = 0$$

$$\sin\theta = t \quad \text{③}$$

$$8t^2 - 4t - 3 = 0 \text{ の解を } t$$

$$t = \frac{1 \pm \sqrt{7}}{4}$$

$$0 < \theta < 180^\circ \text{ より } t > 0$$

$$\sin\theta = \frac{1 + \sqrt{7}}{4} \quad \text{④}$$

②より

$$\cos\theta = \frac{1}{2} - \frac{1 + \sqrt{7}}{4} \quad \text{⑤}$$

$$\cos\theta = \frac{1 - \sqrt{7}}{4} \quad \text{⑥}$$

$$\begin{aligned} \tan\theta &= \frac{1}{\tan\theta} \\ &= \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} \\ &= \frac{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}{\sin\theta\cos\theta} \quad \text{⑦} \end{aligned}$$

①, ②, ④, ⑤, ⑦より

$$\frac{\frac{1}{2} \cdot \frac{\sqrt{7}}{2}}{-\frac{3}{8}} = -\frac{2\sqrt{7}}{3}$$